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Stegall compact spaces which are not fragmentable [☆]

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Abstract

Using modifications of the well-known construction of "double-arrow" space we give consistent examples of nonfragmentable compact Hausdorff spaces which belong to Stegall's class $\mathcal{S}$. Namely the following is proved.

(1) If $\aleph_1$ is less than the least inaccessible cardinal in L and MA & $\neg$CH hold then there is a nonfragmentable compact Hausdorff space K such that every minimal usco mapping of a Baire space into K is singlevalued at points of a residual set.

(2) If $V = L$ then there is a nonfragmentable compact Hausdorff space K such that every minimal usco mapping of a completely regular Baire space into K is singlevalued at points of a residual set.

Keywords: Fragmentable compact space; Stegall's class of compact spaces; Minimal usco mapping

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0. Introduction

The class $\mathcal{S}$ of topological spaces, introduced by Stegall in [11], plays an important role in the study of Gâteaux differentiability of convex functions on Banach spaces. Stegall proved that, if the dual unit ball $B_{X^\star}$ of a Banach space X, equipped with the $w^\star$ topology, belongs to the class $\mathcal{S}$, then every convex continuous function on X is Gâteaux differentiable at points of a residual subset of X (i.e., X is a *weak Asplund space*). Moreover, the class of all Banach spaces whose dual ball belongs to $\mathcal{S}$ has surprisingly good permanence properties, which are not known for weak Asplund spaces. Up to now it also remains open whether these two classes coincide. Almost everything which is known about differentiability of convex functions can be found in a recent book of Fabian [1].

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There is a subclass of $\mathcal{S}$ which occurs widely in the Banach space theory. This is the class of all fragmentable spaces. In this context there is a theorem of Ribarska [10] which states that if X has a Gâteaux differentiable norm then the dual unit ball $B_{X^\star}$ is fragmentable. Ribarska [9] also proved that if a compact Hausdorff space K is fragmentable, the same holds for $(B_{\mathcal{C}(K)^\star}, w^\star)$. The analogous question about the class $\mathcal{S}$ is open. We will show consistent examples of nonfragmentable compact Hausdorff spaces which belong to $\mathcal{S}$. Our construction is based on modifying the well-known example of "double arrow" space and on study of special subsets of $\mathbb{R}$. The main result is contained in the point (2) of theorem. We remark also that we get a slightly weaker example (point (3) of theorem) by a rather direct use of a lemma of Namioka and Pol [8]. However it remains open whether the dual unit ball of the space of continuous functions on some of here constructed compact spaces endowed with the $w^\star$ topology belongs to $\mathcal{S}$. First let us give the definitions.

The topological space Y is *fragmented* by a metric ρ if any nonempty subset of Y contains a nonempty relatively open set of arbitrarily small (ρ-)diameter. The space Y is said *fragmentable* if it is fragmented by some metric.

The class $\mathcal{S}$ is defined via minimal usco mappings. So we recall the definition of such mappings.

Let $\varphi: X \to Y$ be a set-valued mapping acting between two Hausdorff topological spaces. We say that φ is an *usco mapping* (*upper semicontinuous compact-valued*) if, for every $x \in X$ $\varphi(x)$ is a nonempty compact subset of Y, and whenever $\varphi(x) \subset U$ where U is open in Y, there is a neighborhood V of x such that $\varphi(V) \subset U$ (or, equivalently,

$$\varphi^{-1}(F) = \{x \in X \mid \varphi(x) \cap F \neq \emptyset\}$$

is closed in X whenever F is closed in Y). An usco mapping $\varphi: X \to Y$ is called *minimal* if it is minimal with respect to the inclusion (i.e., if $\psi: X \to Y$ is usco such that $\psi(x) \subset \varphi(x)$ for every $x \in X$ then $\psi = \varphi$).

We will denote by $\mathcal{B}$ the class of all (Hausdorff) Baire spaces. If $\mathcal{C}$ is a subclass of $\mathcal{B}$, and Y a (Hausdorff) space, we say that Y is a *Stegall space with respect to* $\mathcal{C}$ (we write $Y \in \mathcal{S}(\mathcal{C})$) if, for every $X \in \mathcal{C}$ and for every minimal usco mapping $\varphi: X \to Y$, the mapping φ is singlevalued at least at one point. The topological space Y belongs to *Stegall's class* $\mathcal{S}$ if it belongs to $\mathcal{S}(\mathcal{B})$.

We will need some properties of minimal usco mappings which are summed up in following lemmas.

Lemma 1 [12, p. 74]. *Let $\varphi: X \to Y$ be an usco mapping. Then the following assertions are equivalent*:

(a) *φ is a minimal usco mapping*;
(b) *whenever $U \subset X$ and $W \subset Y$ are open sets such that $\varphi(U) \cap W \neq \emptyset$ then there is a nonempty open $V \subset U$ such that $\varphi(V) \subset W$*;
(c) *whenever $U \subset X$ is open and $F \subset Y$ closed such that $\varphi(x) \cap F \neq \emptyset$ for every $x \in U$ then $\varphi(U) \subset F$*;
(d) *for any topological space Z and $f: Y \to Z$ continuous* (*single-valued*) *map the composition $f \circ \varphi$ is a minimal usco.*

Lemma 2. *Let $\varphi : X \to Y$ be a minimal usco mapping. If $A \subset X$ is open or dense then $\varphi \restriction A$ is a minimal usco mapping.*

Proof. Clearly $\varphi \restriction A$ is usco. To see the minimality let $W \subset Y$ and $U \subset A$ be relatively open such that $\varphi(U) \cap W \neq \emptyset$. If A is open then U is open in X, hence by minimality of φ there is $V \subset U$ nonempty open such that $\varphi(V) \subset W$. If A is dense, let U' be open in X such that $U' \cap A = U$. Then $\varphi(U') \cap W \neq \emptyset$, so by minimality of φ, there is $V' \subset U'$ nonempty open with $\varphi(V') \subset W$. By density of A we have that $V = V' \cap A$ is a nonempty relatively open subset of U such that $\varphi(V) \subset W$. Hence, by Lemma 1, $\varphi \restriction A$ is minimal. $\square$

Lemma 3. *Let $\varphi : X \to Y$ be an usco mapping such that the set $\{x \in X \mid \varphi(x)$ is a singleton$\}$ has empty interior. Then the following two conditions are equivalent*:

(i) *φ is minimal*;

(ii) *for every $G \subset X$ nonempty open and every choice $y_x \in \varphi(x)$ the set $\{y_x \mid x \in G\}$ has no isolated points.*

Proof. (ii) $\Rightarrow$ (i) We will use the characterization of minimal usco mappings given in Lemma 1(b). Let $U \subset X$ and $W \subset Y$ be open such that $\varphi(U) \cap W \neq \emptyset$. Put $F = \{x \in U \mid \varphi(x) \setminus W \neq \emptyset\}$. Since φ is usco, F is relatively closed in U. If $V = U \setminus F \neq \emptyset$ then V is a nonempty open subset of U satisfying $\varphi(V) \subset W$. If $U = F$, choose $x_0 \in U$ and $y_{x_0} \in \varphi(x_0) \cap W$. This is possible since $\varphi(U) \cap W \neq \emptyset$. For $x \in U \setminus \{x_0\}$ let $y_x \in \varphi(x) \setminus W$. Then $\{y_x \mid x \in U\} \cap W = \{y_{x_0}\}$, so y_{x_0} is an isolated point, a contradiction.

(i) $\Rightarrow$ (ii) Suppose φ is minimal, let $G \subset X$ be nonempty open, $y_x \in \varphi(x),\ x \in G$ be an arbitrary choice. If $\{y_x \mid x \in G\}$ has an isolated point, it means that there is $x_0 \in G$ and $W \subset Y$ open such that $W \cap \{y_x \mid x \in G\} = \{y_{x_0}\}$. So $\varphi(G) \cap W \neq \emptyset$, hence by Lemma 1(b) we get a nonempty open $V \subset G$ with $\varphi(V) \subset W$. Now, for $x \in V$ we have $y_x \in W$, therefore $y_x = y_{x_0}$ and thus $y_{x_0} \in \varphi(x)$. It follows by Lemma 1(c) that $\varphi(V) \subset \{y_{x_0}\}$, which is a contradiction with the assumptions on φ. $\square$

The statement of Lemma 2 enables us to prove the following localization statement.

Proposition 1. *Let $\mathcal{C}$ be a subclass of $\mathcal{B}$ which is closed with respect to open subspaces and Y be a topological space. Then the following assertions are equivalent*:

(i) *$Y \in \mathcal{S}(\mathcal{C})$.*

(ii) *For every $X \in \mathcal{C}$ and every minimal usco $\varphi : X \to Y$ the mapping φ is single-valued at points of a dense subset of X.*

If $\mathcal{C}$ is moreover closed with respect to dense $\mathcal{G}_\delta$-subspaces then the conditions (i) *and* (ii) *are also equivalent with the following ones.*

(iii) *For every $X \in \mathcal{C}$ and every minimal usco $\varphi : X \to Y$ the mapping φ is single-valued at points of a subset of X of the second category.*

(iv) *For every $X \in \mathcal{C}$ and every minimal usco $\varphi : X \to Y$ the mapping φ is single-valued at points of a dense Baire subspace of X.*

If $\mathcal{S}$ is closed with respect to dense Baire subspaces then the previous conditions are equivalent to the condition:

(v) *For every $X \in \mathcal{C}$ and every minimal usco $\varphi : X \to Y$ the mapping φ is single-valued at points of a residual subset of X.*

Proof. This is an easy consequence of Lemma 2 and Banach localization principle. □

Let us give some examples of classes $\mathcal{C}$. For example, the class of all Baire spaces, or of all completely regular Baire spaces, of Baire spaces with pseudoweight $\leqslant \aleph_1$, or of all ccc Baire spaces are closed with respect to open subsets and dense Baire subspaces. The class of complete metric spaces, and that of (almost) Čech complete spaces, or that of those spaces X for which $X \times X$ is a Baire space, are closed to open subsets and dense $\mathcal{G}_\delta$-subsets.

We will need also a property of fragmentable compact spaces.

Lemma 4. *Let K be a compact Hausdorff space. If K is fragmentable then the topology of K has a σ-scattered network* (*a network is such a family $\mathcal{N}$ of subsets of K such that whenever $G \subset K$ is open and $x \in G$ then there is $N \in \mathcal{N}$ such that $x \in N \subset G$*).

Proof. If K is fragmentable then, by [9], there exist a sequence $\mathcal{U}_n$ of scattered covers of K, such that the family $\mathcal{U} = \bigcup_{n\in\mathbb{N}} \mathcal{U}_n$ separates points of K and for each n the collection $\{\overline{U} \mid U \in \mathcal{U}_{n+1}\}$ is a refinement of $\mathcal{U}_n$. Then $\mathcal{U}$ is a σ-scattered network of K. To see it let $G \subset K$ be open and $x \in G$. For every n there is a unique $U_n \in \mathcal{U}_n$ with $x \in U_n$. Since $\mathcal{U}$ separates points, we get $\{x\} = \bigcap_{n\in\mathbb{N}} U_n$, and using the fact that $\overline{U_{n+1}} \subset U_n$ we conclude that $\{x\} = \bigcap_{n\in\mathbb{N}} \overline{U_n}$. It follows, by compactness of $\overline{U_n}$, that $U_n \subset G$ for some n. This finishes the proof that $\mathcal{U}$ is a network. □

1. The spaces K_A

We will construct some examples of compact spaces which are not fragmentable but under some set theoretical assumptions belong to the class $\mathcal{S}$. This is a generalized construction of the “double arrow space”, which is our space K_A for $A = (0,1)$ and $K = [0,1]$.

First we need some definitions. If $A \subset \mathbb{R}$ is a set without isolated points we say that $x \in A$ is *left-isolated point* of A if there is $\varepsilon > 0$ such that $(x - \varepsilon, x) \cap A = \emptyset$. The set of all left-isolated points of A we will denote by A^l. Similarly we define *right-isolated points* and A^r. We put $A^i = A^l \cup A^r$ and $A^d = A \setminus A^i$. It is easy to see that A^i is countable, and that in the case when A is closed (without isolated points) we have $(A^d)^d = A^d$ (in fact this equality holds whenever A is locally uncountable, that is, every nonempty relatively open subset is uncountable).

Let $K \subset \mathbb{R}$ be a compact perfect set, $A \subset K^d$ be arbitrary. Put $K_A = ((K^l \cup A) \times \{1\}) \cup ((K \setminus K^l) \times \{0\})$. We define on K_A a topology—the neighborhood basis of $(t, \varepsilon) \in K_A$ will be

(i) $\{(t - \Delta, t + \Delta) \times \{0, 1\} \cap K_A \mid \Delta > 0\}$ if $t \in K^d \setminus A$ (and $\varepsilon = 0$);
(ii) $\{((t - \Delta, t] \times \{0\} \cup (t - \Delta, t) \times \{1\}) \cap K_A \mid \Delta > 0\}$ if $t \in A \cup K^r$ and $\varepsilon = 0$;
(iii) $\{((t, t + \Delta) \times \{0\} \cup [t, t + \Delta) \times \{1\}) \cap K_A \mid \Delta > 0\}$ if $t \in A \cup K^l$ and $\varepsilon = 1$.

Proposition 2. *Let $K \subset \mathbb{R}$ be a perfect compact set, $A \subset K^d$ be arbitrary. Let K_A be as above, with the topology defined by* (i)–(iii). *Then K_A is a first countable hereditarily Lindelöf and hereditarily separable compact Hausdorff space.*

Proof. It is obvious that the topology of K_A is Hausdorff and first countable. Next we will prove that K_A is compact. Let $\mathcal{U}$ be an open cover of K_A. Then for each $x \in K_A$ there is $U_x \in \mathcal{U}$ such that $x \in U_x$. Now let $t \in K$ be arbitrary. If $t \notin A$ then there is $\varepsilon_t > 0$ satisfying $(t - \varepsilon_t, t + \varepsilon_t) \times \{0, 1\} \cap K_A \subset U_{(t,0)}$. If $t \in A$, there is $\varepsilon_t > 0$ such that

$$\big((t, t + \varepsilon_t) \times \{0\} \cup [t, t + \varepsilon_t) \times \{1\}\big) \cap K_A \subset U_{(t,1)} \quad \text{and}$$
$$\big((t - \varepsilon_t, t] \times \{0\} \cup (t - \varepsilon_t, t) \times \{1\}\big) \cap K_A \subset U_{(t,0)}.$$

Now, the intervals $(t - \varepsilon_t, t + \varepsilon_t)$, $t \in K$ cover K and since this one is compact, there is $F \subset K$ finite such that $(t - \varepsilon_t, t + \varepsilon_t)$, $t \in F$ cover K. Then clearly $U_{(t,0)}$, $t \in F$ and $U_{(t,1)}$, $t \in F \cap A$ form a finite subcover of $\mathcal{U}$.

Let us prove now that K_A is hereditarily Lindelöf. Let $\mathcal{U}$ be a family of open sets in K_A. Put $G = \bigcup \mathcal{U}$. First we prove that the set

$$S = \big\{t \in A \mid G \text{ contains exactly one of the points } (t, 0),\ (t, 1)\big\}$$

is countable. Let

$$S_i = \big\{t \in A \mid (t, i) \in G \text{ and } (t, 1 - i) \notin G\big\} \quad \text{for } i = 0, 1.$$

For $t \in S_0$ choose $\Delta_t > 0$ such that

$$\big((t - \Delta_t, t] \times \{0\} \cup (t - \Delta_t, t) \times \{1\}\big) \cap K_A \subset G.$$

Then the intervals $(t - \Delta_t, t]$, $t \in S_0$ are disjoint ad hence S_0 is countable. Similarly S_1 is countable, and so $S = S_0 \cup S_1$ is countable. For any $x \in G \setminus S$ let $U_x \in \mathcal{U}$ be such that $x \in U_x$. Now, for $t \in K$ such that $\{(t, 0), (t, 1)\} \cap K_A \subset G$ we choose $\varepsilon_t > 0$ in the same way in the previous paragraph. By the same argument, using the fact that K is hereditarily Lindelöf, we obtain a countable subfamily $\mathcal{U}_1 \subset \mathcal{U}$ covering $G \setminus S \times \{0, 1\}$. Since S is countable, there is a countable subfamily of $\mathcal{U}$ covering G.

Finally we will prove that K_A is hereditarily separable. Let $M \subset K_A$ be arbitrary, $N = \{t \in K \mid \{(t, 0), (t, 1)\} \cap M \neq \emptyset\}$. Let C be a countable dense subset of N^d. Then $(N^i \cup C) \times \{0, 1\} \cap M$ is a countable dense subset of M. $\square$

In the following three propositions we give the characterization of those spaces among K_A's which are fragmentable, which belong to $\mathcal{S}(\mathcal{C})$, and which satisfy a condition which is necessary for a compact space to belong to $\mathcal{S}$ (the condition (i) in Proposition 5).

Proposition 3. *Let $K \subset \mathbb{R}$ be a compact perfect set and $A \subset K^d$ be arbitrary. Then the following assertions are equivalent*:

(a) *A is countable*;
(b) *K_A is metrizable*;
(c) *K_A is fragmentable.*

Proposition 4. *Let $\mathcal{C}$ be a subclass of the class of Baire spaces closed with respect to open subspaces and dense $\mathcal{G}_\delta$ subspaces. Let $K \subset \mathbb{R}$ be a compact perfect set and $A \subset K^d$ be arbitrary. Then the following assertions are equivalent*:

(1) *$K_A \in \mathcal{S}(\mathcal{C})$*;
(2) *For every $X \in \mathcal{C}$ and every continuous $f : X \to A$ there is a nonempty open subset U of X such that the set $f(U)$ has maximum or minimum.*

Proposition 5. *Let $K \subset \mathbb{R}$ be a compact perfect set and $A \subset K^d$ be arbitrary. Then the following conditions are equivalent*:

(i) *Every closed subset of K_A contains a dense completely metrizable subspace.*
(ii) *A is perfectly meager.*

Let us recall that a set $A \subset \mathbb{R}$ is called *perfectly meager*, if every dense in itself subset of A is meager in itself, or, equivalently, if for every perfect set $P \subset \mathbb{R}$ the intersection $P \cap A$ is meager in P. This definition follows [6] where it is also proved that there is an uncountable perfectly meager set.

Now we will prove Proposition 3. To this end we need two lemmas.

Lemma 5. *Let $K \subset \mathbb{R}$ be a perfect compact set, $A \subset K^d$ be arbitrary and $M \subset K_A$. Then M is metrizable if and only if $M \cap (A \times \{0, 1\})$ is countable. In this case M has a countable basis.*

Proof. If $M \cap (A \times \{0, 1\})$ is countable it is easy to see that M has a countable basis, and since M is completely regular (as a subspace of K_A), M is metrizable.

Conversely, suppose M is metrizable. By Proposition 2, the space M is Lindelöf and hence has a countable base $\mathcal{B}$. For each $(t, 0) \in M \cap (A \times \{0\})$ there is $B_t \in \mathcal{B}$ with $(t, 0) \in B_t \subset (t - 1, t] \times \{0\} \cup (t - 1, t) \times \{1\}$. Clearly B_t are different sets for different values of t, so $M \cap (A \times \{0\})$ is countable. Similarly $M \cap (A \times \{1\})$ is countable which completes the proof. □

Lemma 6. *Let X be a topological space. Then the following conditions are equivalent*:

(a) *X is hereditarily Lindelöf*;
(b) *every scattered family of subsets of X is countable*;
(c) *every scattered subset of X is countable.*

Proof. (a) $\Rightarrow$ (b) Let $\mathcal{U}$ be an uncountable scattered family in X. Then there is a well-ordering $(U_\alpha \mid \alpha < \Gamma)$ of $\mathcal{U}$ such that Γ is an uncountable ordinal, and for any $\alpha < \Gamma$ the set $\bigcup_{\beta<\alpha} U_\beta$ is relatively open in $\bigcup \mathcal{U}$. Put $Y_\beta = \bigcup_{\alpha<\beta} U_\alpha$ for $\beta \leqslant \Gamma$. Then $(Y_\beta \mid \beta < \omega_1)$ is an open cover of Y_{ω_1} without countable subcover.

The implication (b) $\Rightarrow$ (c) is obvious.

(c) $\Rightarrow$ (a) Let $\mathcal{U}$ be a family of open sets in X such that there is no countable subfamily with the same union. By transfinite induction we find, for $\alpha < \omega_1$, $U_\alpha \in \mathcal{U}$ and $x_\alpha \in X$ such that $x_\alpha \in U_\alpha \setminus \bigcup_{\beta<\alpha} U_\beta$. Then $\{x_\alpha \mid \alpha < \omega_1\}$ is an uncountable scattered subset of X. $\square$

Proof of Proposition 3. The equivalence (a) $\Leftrightarrow$ (b) follows immediately from Lemma 5.

The implication (b) $\Rightarrow$ (c) follows from the definition of fragmentability.

(c) $\Rightarrow$ (a) Let A be uncountable and $\mathcal{N}$ be a network of K_A. By the same method as in the proof of Lemma 5 we show that $\mathcal{N}$ is uncountable and therefore is not σ-scattered (since, by Lemma 6, in hereditarily Lindelöf space every scattered family is countable), so by Lemma 4 the space K_A is not fragmentable. $\square$

To prove Propositions 4 and 5 we need some finer properties of the spaces K_A which are collected in the following proposition.

Proposition 6. *Let $K \subset \mathbb{R}$ be a compact perfect set and $A \subset B \subset K^d$. Let us define $F : K_B \to K_A$ by the formula*

$$F(t,\varepsilon) = \begin{cases} (t,0), & t \in B \setminus A, \\ (t,\varepsilon) & \text{otherwise}. \end{cases}$$

Then the following holds.

(1) *F is continuous and F^{-1} is a minimal usco mapping.*
(2) *If $G \subset K_B$ has nonempty interior in K_B then $F(G)$ has nonempty interior in K_A.*
(3) *If $M \subset K_B$ is a Borel set then $F(M)$ is Borel and $F^{-1}(F(M)) \setminus M$ is countable.*
(4) *If $M \subset K_B$ then M is nowhere dense* (*respectively meager*) *in K_B if and only if $F(M)$ is nowhere dense* (*respectively meager*) *in K_A.*
(5) *If X is a Baire space and $\varphi : X \to K_B$ is a minimal usco mapping then there is a residual set $G \subset X$ such that for each $x \in G$ we have $\varphi(x) \subset \{t\} \times \{0,1\}$ for some $t \in K$.*
(6) *If X is a topological space and $f : X \to B$ a continuous map, then $F^{-1} \circ f$ is a minimal usco if and only if for every open $G \subset X$ we have $(f(G))^i = \emptyset$.*

Proof. (1) The continuity of F follows easily from the definition of the topology of K_A (and K_B). So F^{-1} is usco (by the compactness of K_B). Now, if $U \subset K_A$ and $W \subset K_B$ are open sets such that $F^{-1}(U) \cap W \neq \emptyset$ then $F^{-1}(U) \cap W$ is open in K_B and hence there is some $t \in K_d$ and $\Delta > 0$ such that $(t-\Delta, t+\Delta) \times \{0,1\} \cap K_B \subset F^{-1}(U) \cap W$. Then the set $G = (t-\Delta, t+\Delta) \times \{0,1\} \cap K_A$ is a nonempty open subset of U and $F^{-1}(G) \subset W$. Hence, by Lemma 1, F^{-1} is minimal.

(2) If $G \subset K_B$ has nonempty interior, it follows from the definition of the topology of K_B that there is an open interval (a,b) such that $\emptyset \neq ((a,b)\cap K)\times\{0,1\}\cap K_B \subset G$. Then clearly $\emptyset \neq ((a,b)\cap K)\times\{0,1\}\cap K_A \subset F(G)$, so $F(G)$ has nonempty interior.

(3) Put $\mathcal{M} = \{M \subset K_B \mid F(M)$ is Borel and $F^{-1}(F(M)) \setminus M$ is countable$\}$. We will show that $\mathcal{M}$ is a σ-algebra containing all open sets, and hence it will contain all Borel sets. It is obvious that $\mathcal{M}$ is closed with respect to countable unions. Since $F(F^{-1}(F(M))\setminus M) = F(F^{-1}(F(K_B\setminus M))\setminus(K_B\setminus M))$ and F is at most two-to-one, and since $F(K_B\setminus M) = (K_A\setminus F(M))\cup F(F^{-1}(F(M))\setminus M)$, it follows that $\mathcal{M}$ is closed with respect to complements. The fact that $\mathcal{M}$ contains open sets can be easily seen by an argument used in the proof of Proposition 2.

(4) Let $M \subset K_B$ be nowhere dense. Then $\overline{M}$ is nowhere dense too and $F(\overline{M}) \supset F(M)$, so we can suppose that M is closed. Hence M is compact and thus so is $F(M)$. If $\emptyset \neq G \subset F(M)$ is open in K_A then $F^{-1}(G)$ is open in K_B and, by (3), $F^{-1}(G)\setminus M$ is countable, therefore $F^{-1}(G)\cap M$ is of second category in K_B, which is a contradiction with the assumption that M is nowhere dense. So $F(M)$ is nowhere dense. Now it is obvious that if M is meager then so is $F(M)$.

Conversely suppose that $M \subset K_B$ is such that $F(M)$ is nowhere dense. Then $\overline{F(M)}$ has empty interior in K_A, hence $F^{-1}(\overline{F(M)})$ has empty interior in K_B (by (2)), and therefore M is nowhere dense. Similarly, if $F(M)$ is meager so is M.

(5) Put $A = \emptyset$. Then K_A is canonically homeomorphic to K. Let X be a Baire space and $\varphi: X \to K_B$ a minimal usco mapping. By Lemma 1, $F\circ\varphi$ is a minimal usco mapping (where F is as in (1)), therefore, since K_A is metrizable, $F\circ\varphi$ is singlevalued at points of a residual set $G \subset X$. Now the statement follows immediately from the definition of F.

(6) This follows easily from Lemma 3 and the definition of the topology of K_B. $\square$

Proof of Proposition 4. (1) $\Rightarrow$ (2) Let $X \in \mathcal{C}$ and $f: X \to A$ be a continuous map. If for all nonempty open sets U the image $f(U)$ has neither maximum nor minimum then for every open U we have $f(U)^i = \emptyset$. Suppose this is not the case, i.e., $f(U)^i \neq \emptyset$ for some nonempty open $U \subset X$. Choose $a \in f(U)^i$. Without loss of generality we can suppose that $a \in f(U)^r$. This means that there is $\varepsilon > 0$ such that $[a, a+\varepsilon)\cap f(U) = \{a\}$. Put

$$V = f^{-1}\big((-\infty, a+\varepsilon)\big)\cap U.$$

Then $V \subset X$ is nonempty open and $a = \max f(V)$, a contradiction. Hence, by Proposition 6(6), $F^{-1}\circ f$ is minimal usco (F has the meaning as in Proposition 6—with A in the place of B and with $\emptyset$ in the place of A), which is nowhere singlevalued, which contradicts (1).

(2) $\Rightarrow$ (1) Suppose that $K_A \notin \mathcal{S}(\mathcal{C})$. This means that there is $X \in \mathcal{C}$ and $\varphi: X \to K_A$ which is nowhere singlevalued. Let $F: K_A \to K$ be the canonical surjection. By Proposition 6(5), there is $X_0 \subset X$ dense G_δ (and hence $X_0 \in \mathcal{C}$) such that for every $x \in X_0$ we have $\varphi(x) = \{(t,0),(t,1)\}$ for some $t \in A$. By Lemma 2 the restriction $\varphi \restriction X_0$ is minimal usco. So, by Proposition 6(6), $(F\circ\varphi(U))^i = \emptyset$ for every $U \subset X_0$ nonempty relatively open, in particular for any such U the image $F\circ\varphi(U)$ has neither minimum nor maximum, which contradicts (2). $\square$

Proof of Proposition 5. (i) $\Rightarrow$ (ii) Suppose A is not perfectly meager. It means that there is a perfect set $P \subset K$ such that $P \cap A$ is of second category in P. Then $P^d \cap A$ is of second category in P too. Put

$$H = \big((P^d \times \{0,1\}) \cup \big((P^r \cup (P^l \setminus A)) \times \{0\}\big) \cup ((P^l \cap A) \times \{1\})\big) \cap K_A.$$

Then H is closed in K_A and is canonically homeomorphic to $P_{A \cap P^d}$. By Proposition 6(4), the set $C = (P^d \cap A) \times \{0,1\}$ is of second category in H. If G is a completely metrizable dense subspace of H then $G \cap C$ is of the second category in H, and hence is uncountable. But $G \cap C$ is metrizable and thus, by Lemma 5, countable, which is a contradiction. So H contains no dense completely metrizable subspace.

(ii) $\Rightarrow$ (i) Suppose that A is perfectly meager. Let $H \subset K_A$ be nonempty closed. We can write $H = S \cup P$ with S scattered (hence countable by Lemma 6) and P perfect. Moreover we can suppose $P \cap S = \emptyset$. If $P = \emptyset$ then H is countable compact and therefore metrizable. Now suppose P is nonempty. Let $F : K_A \to K$ be the natural surjection. Then $F(P)$ is a compact perfect subset of K, so $F(P) \cap A$ is meager in $F(P)$. Hence there is $G \subset F(P)$ dense G_δ in $F(P)$ such that $G \cap A = \emptyset$. Then $G \times \{0\}$ is a dense completely metrizable subspace of P. Let S_0 denote the set of all isolated points of S (or, equivalently, of H). Then S_0 is a relatively discrete space and thus completely metrizable (by the discrete metric). The set $G_0 = G \times \{0\} \setminus \overline{S_0}$ is completely metrizable (as a relatively open subspace of $G \times \{0\}$), and so is $S_0 \cup G_0$ (as a topological sum of two completely metrizable spaces). And clearly $S_0 \cup G_0$ is dense in H. □

2. Nonfragmentable compact spaces from $\mathcal{S}$

In this section we collect some examples of uncountable A's which satisfy the condition (2) of Proposition 4 with respect to some classes $\mathcal{C}$. In fact we will consider a stronger condition

(∗) For any $X \in \mathcal{C}$ and any $f : X \to A$ continuous there is $U \subset X$ open such that f is constant on U.

This condition clearly implies the mentioned condition (2), and was considered, with a different purpose in [8].

Proposition 7.

(a) *If A is a $\mathcal{Q}$-set then A satisfies* (∗) *with respect to the class of all Baire spaces with a pseudobase of cardinality $\leqslant \omega_1$.*

(b) *If A is a $\mathcal{Q}$-set and* $\operatorname{card} A = \aleph_1 = \aleph_1^L$ (*or, more generally,* $\operatorname{card} A$ *is less than the least inaccessible cardinal in L*) *then A satisfies* (∗) *with respect to the class of all Baire spaces.*

(c) *If A is a coanalytic set with no perfect subset then A satisfies* (∗) *with respect to the class of all completely regular Baire spaces.*

(d) *If A contains no perfect subset then A satisfies* (∗) *with respect to the class of all* (*almost*) *Čech complete spaces.*

(e) *For any $K \subset \mathbb{R}$ perfect compact set there is $A \subset K$ of cardinality $\aleph_1$ such that A satisfies $(*)$ with respect to the class of spaces X such that $X \times X$ is a Baire space.*

(f) *For any $K \subset \mathbb{R}$ perfect compact set there is $A \subset K$ of cardinality $\aleph_1$ such that A satisfies $(*)$ with respect to the class of all completely regular Baire spaces satisfying ccc.*

The assertion (c) is proved in [8, Lemma 5.1] (let us recall that such an uncountable set exists under $V = L$), the assertions (e), (f) in Section 8 of [8]. The assertion (d) can be proved by a method similar to that used in Section 5 of [8], we will not give here the proof.

To clarify the assertions (a), (b) let us recall that a set $A \subset \mathbb{R}$ is called *a $\mathcal{Q}$-set* if each its subset is relatively $\mathcal{F}_\sigma$. Clearly every countable set is a $\mathcal{Q}$-set. And if we suppose Martin's axiom and the negation of continuum hypothesis there is, by [7, p. 162], an uncountable $\mathcal{Q}$-set.

The proof of (a) follows easily from the following lemma which is a particular case of Theorem 2.4 in [4].

Lemma 7. *Let X be a Baire space of pseudoweight at most $\aleph_1$. Then the union of every disjoint $\mathcal{F}_\sigma$-additive family of meager subsets of X has empty interior.*

Proof. Let $\mathcal{E}$ be an $\mathcal{F}_\sigma$-additive family of meager subsets of X whose union has nonempty interior. Let $G \subset \bigcup \mathcal{E}$ be nonempty open, $(B_\xi,\ \xi < \omega_1)$ be a pseudobasis of the topology of G. It is easy to construct by induction $x_\xi, y_\xi \in G$ and $E_\xi, F_\xi \in \mathcal{E}$ (for $\xi < \omega_1$) such that

(i) $x_\xi \in E_\xi \cap B_\xi,\ y_\xi \in F_\xi \cap B_\xi$,

(ii) $E_\xi \neq F_\xi$,

(iii) $\{E_\xi, F_\xi\} \cap \{E_\eta, F_\eta \mid \eta < \xi\} = \emptyset$.

Now, $C_0 = \bigcup\{E_\xi \mid \xi < \omega_1\} \cap G$ is dense relatively $\mathcal{F}_\sigma$ subset of G, as well as $C_1 = G \setminus C_0$. So, both C_0 and C_1 have empty interior and hence are meager (since they are $\mathcal{F}_\sigma$), therefore their union G is meager, a contradiction, since X is a Baire space. □

Let us prove the assertion (b). Let X be a Baire space, A a $\mathcal{Q}$-set and $f : X \to A$ a continuous map such that the inverse image of every point of A is nowhere dense. Then $f^{-1}(a),\ a \in A$ form an $\mathcal{F}_\sigma$-additive partition of X into nowhere dense sets. So, by lemma in the proof of Theorem 3.3 in [2], we get that for some $\kappa \leqslant \operatorname{card} A$ the cardinal κ is a measurable cardinal in a transitive model M of ZFC. By [14] we get that κ is inaccessible in M, and it is easy to check that κ is inaccessible in L too, a contradiction.

Now let us show that the assumptions of (b) are consistent with ZFC. Let us start with $V = L$. By theorem in Section 7.11. of [13], there is a complete Boolean algebra $\mathcal{B}$ satisfying the countable chain condition, such that for any generic filter G on $\mathcal{B}$ we have

$$V[G] \models \text{ Martin's axiom and the negation of continuum hypothesis.}$$

Since this generic extension is done via a Boolean algebra satisfying the countable chain condition, we get that $\aleph_1^{V[G]} = \aleph_1^V (= \aleph_1^L)$. Now, since in $V[G]$ Martin's axiom and the negation of continuum hypothesis hold, by [7, p. 162] there an uncountable $\mathcal{Q}$-set A_0. Let $A \subset A_0$ be any set of cardinality $\aleph_1$. Then this set satisfies the assumptions of (b).

Theorem.

(1) *Assume Martin's axiom and the negation of continuum hypothesis. Then there is a Hausdorff compact space K which is not fragmentable but is Stegall with respect to the class of Baire spaces of pseudoweight at most $\aleph_1$.*

(2) *Assume Martin's axiom and the negation of continuum hypothesis, and moreover $\aleph_1 = \aleph_1^L$ (or, more generally, $\aleph_1$ is less than the least inaccessible cardinal in L). Then there is a Hausdorff compact space K which is not fragmentable but belongs to $\mathcal{S}$.*

(3) *Assume $V = L$. Then there is a Hausdorff compact space K which is not fragmentable but is Stegall with respect to the class of all completely regular Baire spaces.*

Proof. This follows immediately from Propositions 7 and 4. $\square$

Let us remark that using only the property $(*)$ we cannot get an absolute example of a Stegall nonfragmentable compact space among spaces K_A, namely the following holds (for the definition of a precipitous ideal over ω_1, see, e.g., [2], let us recall that its existence is equiconsistent with the existence of a measurable cardinal).

Proposition 8. *Suppose there is a precipitous ideal over ω_1. Then for every uncountable separable metric space B there is a Baire metric space X of weight $\leqslant 2^{\omega_1}$ and a continuous function $f : X \to M$ such that $f^{-1}(m)$ is nowhere dense in X for every $m \in M$.*

Proof. If there is a precipitous ideal over ω_1 then, by [2, Theorem 3.2] there is a Baire metric space Y of weight $\leqslant 2^{\omega_1}$ and a partition $(Y_\xi)_{\xi<\omega_1}$ of Y into nowhere dense sets, such that $\bigcup_{\xi\in A} Y_\xi$ has the Baire property in Y for every $A \subset \omega_1$. Now, let M be an uncountable separable metric space. Let $\varphi : \omega_1 \to M$ be a one-to-one map. We define $f : Y \to M$ by the formula $f(x) = \varphi(\xi)$ for $x \in Y_\xi$. Clearly the inverse image of any subset of M has the Baire property in Y, so in particular f has the Baire property. By [5] there is $X \subset Y$ dense G_δ such that $f \restriction X$ is continuous. And clearly the inverse image of every point of M is nowhere dense in X. $\square$

Notice also that if it is consistent to suppose that there is a measurable cardinal, then it is consistent to suppose that there is a precipitous ideal over ω_1 and Martin's axiom and the negation of continuum hypothesis hold (this follows from [3]), so in Proposition 7(a) the assumption on pseudoweight cannot be dropped.

But it remains open whether there is an absolute example of an uncountable set $A \subset \mathbb{R}$ satisfying the condition (2) of Proposition 4, it is even possible that this condition is satisfied by every perfectly meager set A. Another question is what we can say about $(B_{\mathcal{C}(K_A)^\star}, w^\star)$ for our sets A.

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